SORTING NETWORKS

The Gap Is the Coupling

Order and Parity as Two Projections of One Relation

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Abstract

A discharge calculus for comparator networks maintains two ledgers: an **ordering** state recording which wire pairs are settled, and a **parity** state recording which comparator branches fired. They are not two structures requiring combination: they are two projections of a single relation, and the arithmetic gap between the relation and the product of its projections is exactly the information the coupling carries.

Concretely, let T be the set of pairs (final configuration, branch-atom vector) realised by a network. Measured on optimal networks at $n = 4, 5, 6$, $|T|$ is 16, 32, 64 while $|\pi_1 T| \cdot |\pi_2 T|$ is 60, 162, 336. No functional dependency holds in either direction, so neither ledger determines the other.

The coupling is Iyun's parity–direction lock applied **conditionally**. Unconditionally the lock forces nothing — both branches remain possible at every step — while under a context of earlier atoms it forces the next branch in 11 of 11 cases at $n = 4$ and 206 of 206 at $n = 6$, without exception. The fraction of forced branches rises monotonically along the network ($0.00 \to 0.50 \to 0.72 \to 0.94$) while the absolute number of **free** contexts stays between two and six. The parity ledger's genuine freedom does not grow with the network; only its determined part does.

The realisable atom set is not an affine subspace over $\mathrm{GF}(2)$, and comparators sharing a wire do not exclude one another. Exclusivity is correlated with the role of the shared wire and not determined by it, so no local law on pairs of comparators captures the coupling.

1 Two ledgers, or one?

A discharge calculus for comparator networks carries two kinds of fact. The **ordering** ledger records pairs (i, j) for which $w_i \leq w_j$ has become settled. The **parity** ledger records, for each comparator, which of its two branches fired — a **branch atom** — and facts derived under conjunctions of those atoms are conditional on them.

The two ledgers are not independent structures combined by coupling rules. **There is one structure, and the ledgers are two projections of it.** The coupling rules are not additional axioms; they are the

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identification made explicit.

2 The object

Fix a comparator network $C = (c_0, \ldots, c_{m-1})$ on n wires. For a zero–one input v, running C produces both a final configuration and a vector of branch outcomes. Collect the pairs:

$$T = \{ (\text{final}(v), \text{ atoms}(v)) : v \in \{0,1\}^n \}.$$

Write $S = \pi_1 T$ for the reachable set and $A = \pi_2 T$ for the realisable atom vectors. The ordering ledger is an approximation to S; the parity ledger is an approximation to A.

Proposition 1 (T is not a product). *$|T| < |S| \cdot |A|$, strictly, on every network measured.*

network	$\|T\|$	$\|S\|$	$\|A\|$	$\|S\| \cdot \|A\|$
n4_optimal	16	5	12	60
n5_classic_9	32	6	27	162
n6_dobbelaere_12	64	7	48	336

The gap is the coupling. If the ledgers were independent, T would be the full product and each could be maintained in ignorance of the other. It is not, and the deficit measures precisely how much the pairing constrains beyond what either projection says alone.

Proposition 2 (Neither ledger determines the other). *None of the three maps*

$$\text{atoms} \mapsto \text{state}, \qquad \text{state} \mapsto \text{atoms}, \qquad \text{atoms} \mapsto \text{input}$$

is functional, on any network measured ($n = 4, 5, 6, 8$).

So the situation is not that one ledger is redundant. Both are genuinely partial, and the missing information lives in the pairing.

3 The coupling is the lock, conditionally

Iyun's parity–direction lock [1] identifies a branch atom with an ordering assertion: for a comparator $c = (i, j)$ and configuration v,

$$O_c(v) \iff (v_i \le v_j), \qquad t_c(v) \iff (v_j \le v_i),$$

the even branch **being** the forward assertion and the odd branch the reversed one. They are, in Iyun's phrase, "one fact wearing two names."

Applied unconditionally, this identification forces nothing.

Proposition 3 (The lock is vacuous unconditionally). *At every step of `n4_optimal`, both branches of the next comparator are realisable from the current reachable set. The ordering state alone never determines a branch atom.*

The force appears only under a context. Let h be a conjunction of earlier branch atoms. By the lock, h carries a set of **directed** ordering facts — one per atom, at the time that comparator ran. Those facts may entail the direction of the next comparator, and when they do, the next atom is determined.

Theorem 1 (Conditional forcing is complete)**.** *Every forced branch is entailed by the ordering state induced by its context. On `n4_optimal`, 11 contexts force a branch and all 11 are so explained; on `n6_dobbelaere_12`, 206 of 206.*

A representative instance, the first at $n = 4$ with $C = \langle (0,1), (2,3), (0,2), (1,3), (1,2) \rangle$:

$$\begin{array}{l} g_0 = 1 \implies w_1 \le w_0 \quad (\text{op } 0 = (0,1),\ \text{reversed}) \\ g_1 = 0 \implies w_2 \le w_3 \quad (\text{op } 1 = (2,3),\ \text{forward}) \\ \hline \text{forces } g_2 = 0,\ \text{i.e. } w_0 \le w_2 \text{ at op } 2 = (0,2). \end{array}$$

Neither premise mentions the pair $(0,2)$. The conclusion follows because the two lock-facts compose with the ordering already present.

4 Freedom does not grow

The quantitative shape of the composition is the part with practical consequence. Let the **forcing fraction** at step k be the proportion of realisable contexts of length k under which the next atom is determined.

step k	1	2	3	5	8	11
`n4_optimal`	0.00	0.50	0.50	—	—	—
`n6_dobbelaere_12`	0.00	0.00	0.50	0.72	**0.94**	0.93

The fraction rises monotonically. More striking is the absolute count: the number of **free** contexts — those under which both branches remain possible — stays between two and six across the whole of `n6_dobbelaere_12`, while the forced count grows from 0 to 42.

Corollary 1 (Why the calculus stays small)**.** *The parity ledger's genuine branching does not grow with the network. What grows is the determined part, which by Theorem 1 is already implied by the ordering side.*

This is the structural reason a two-ledger calculus is tractable where unrestricted branch enumeration is not. Most apparent branching is order already decided, recorded on the other ledger.

5 No local exclusion law

Branch atoms are $\mathrm{GF}(2)$ quantities and the irreducible terms of the record are XOR relations, but the realisable atom set is not an affine subspace over $\mathrm{GF}(2)$: an exhaustive search over linear relations on `n4_optimal` finds none holding across A.

Nor do two comparators sharing a wire exclude one another, though two reversals on a shared wire assert opposing directions on it. The failure is graded:

network	same role pairs	same role exclusive	different role pairs	different role exclusive
n4_optimal	4	4	4	0
n5_classic_9	14	14	10	1
n6_classic_12	20	18	18	1
n6_dobbelaere_12	18	14	18	2
n7_dobbelaere_16	27	19	30	5

Here **same role** means the shared wire is the minimum output of both comparators, or the maximum of both. That refinement is exact at $n = 4, 5$ and then degrades — 18 of 20, 14 of 18, 19 of 27. So exclusivity is **correlated** with shared-wire role and not determined by it.

The negative is worth stating plainly: **the coupling is not captured by any local law on pairs of comparators.** Exclusions arise from composition along the network, which is consistent with Theorem 1 — forcing is a property of a whole context, not of a pair.

6 Consequence for discharge

The two ledgers compose with order for a reason that is now visible. The ordering ledger holds facts that are unconditionally true; the parity ledger holds facts conditional on branches; and each branch, by the lock, **is** an ordering fact at its time. **Discharge** — establishing a fact on both halves of a branch and concluding it unconditionally — is the operation that converts the second into the first.

An approximation carrying **only** ordering facts, with no branch atoms, therefore cannot discharge: it has nothing to split on. Its transfer rules are sound implications rather than equivalences, and it is incomplete in exactly the cases where a fact becomes true without any ordering premise having held.

In an atom-free propagator the four rewrite rules

$$(a,x)' \Leftarrow (a,x) \vee (b,x), \qquad (x,b)' \Leftarrow (x,a) \vee (x,b),$$
$$(x,a)' \iff (x,a) \wedge (x,b), \qquad (b,x)' \iff (a,x) \wedge (b,x)$$

hold as implications; the two **disjunctive** rules lose their converse while the two conjunctive ones do not. The minimal witness is $n = 4$ with four comparators, and no instance exists at $n = 3$ at all.

7 Provenance and status

The framing — that composing the parity state with the ordering state is "intersecting two views on the same object" — is Heath Hunnicutt's. The parity–direction lock is Iyun's [1], on Hunnicutt's observation. The contribution here is the measurement: that the product deficit is the coupling, that conditional forcing accounts for every forced branch without exception, and that free contexts do not grow. The shared-wire conjecture is recorded as refuted.

Novelty is not established. We have not surveyed the literature on combining abstract domains, where reduced products and logical products treat closely related questions, nor the sorting-network literature systematically. The results above are reported as measured, with reproducible tables, and should be checked against prior work before any priority is claimed.

Every table is reproducible by exhaustive enumeration over 2^n inputs, held in a quarantined module so that no verifier can reach it.

References

[1] Iyun (Ruach Tov Collective). **The Bishop: Permutation Parity is the Ordering Direction**. July 2026. Machine-checked in Isabelle/HOL.

[2] Burton J. Smith. **An Analysis of Sorting Networks**. MIT Project MAC Technical Report MAC TR-105, October 1972.

[3] Robert W. Floyd and Donald E. Knuth. **The Bose–Nelson Sorting Problem**. Stanford Computer Science Report STAN-CS-70-177, November 1970. (The 0–1 principle appears as Corollary 1.)

[4] Donald E. Knuth. **The Art of Computer Programming, Volume 3: Sorting and Searching**, 2nd ed. Addison-Wesley, 1998. §5.3.4.