The Bishop: Permutation Parity is the Ordering Direction

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Abstract

We prove a small structural theorem about sorting networks that turns out to say something about what sorting **is**. When the discharge calculus verifies a comparator network, it records **branch atoms**: for each comparator, whether it swapped (the **odd** branch) or not (the **even** branch). These branch atoms carry an affine, $\mathrm{GF}(2)$/XOR structure that a purely order-theoretic (monotone) account cannot capture. We show that this parity structure is not independent of the order it builds: **the parity of a branch is welded to the direction of the ordering relation it asserts**. The even branch establishes $a \leq b$; the odd branch establishes $b \leq a$. Like a bishop confined to one colour of a chessboard, a branch cannot assert a direction of the opposite colour to its parity, and no comparator moves it off its colour. The result is machine-checked in Isabelle/HOL. Its portent: the **algebraic** structure of sorting (the parity of the transpositions performed) and the **order-theoretic** structure of sorting (the $\leq$ relation produced) are not two things layered together — they are the same distinction, viewed twice.

1 Preliminaries

We model a configuration as a map from wire indices to a linear order, a comparator as an ordered pair $c = (i, j)$, and `apply_comp` as the operation that places the minimum of wires i, j on wire i and the maximum on wire j. A comparator has exactly two **branches** on a given input, and this is the whole of the structure we need.

The **even branch** O_c ("no-swap") fires when the input already satisfies $v_i \leq v_j$: the comparator leaves the values in place. The **odd branch** t_c ("swap") fires when $v_j < v_i$: the comparator exchanges them. We call these the two **parities** of the comparator, because a swap is a transposition (an odd permutation) and a no-swap is the identity on the pair (an even permutation).

The discharge calculus records, for each comparator, which branch fired, as a **branch atom**. Conjunctions of branch atoms form **contexts**, and facts derived under a context are conditional on those branches. The irreducible terms of this record are XOR relations among branch outcomes — e.g. $(O_c \oplus O_d) \Rightarrow O_e$ — and it is exactly these XOR terms that a monotone (Horn, or median/2-SAT) account of the reachable set cannot express, because parity is the canonical non-monotone relation. The branch atoms live in an **affine** layer over $\mathrm{GF}(2)$.

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2 The lock

Write $O_c(v)$ for "the even branch fires on input v" and $t_c(v)$ for "the odd branch fires." By definition, for $c = (i, j)$,

$$O_c(v) \;=\; \big(v_i \le v_j\big), \qquad t_c(v) \;=\; \big(v_j \le v_i\big).$$

Let $\mathrm{fwd}_c(v) = (v_i \le v_j)$ and $\mathrm{rev}_c(v) = (v_j \le v_i)$ be the two directions the comparator's pair can be ordered. The theorem is immediate — and that is the point.

Theorem 1 (Parity–direction lock)**.** *For every comparator c and configuration v,*

$$O_c(v) \iff \mathrm{fwd}_c(v), \qquad t_c(v) \iff \mathrm{rev}_c(v).$$

*The even branch **is** the forward assertion; the odd branch **is** the reversed assertion.*

The proof is by the definitions above; there is nothing to compute. The content is not in the difficulty but in the **identification**: the branch's parity and the direction of the ordering fact it asserts are the same proposition. They were never two facts that happen to correlate; they are one fact wearing two names.

The diagonal. There is one honest nuance, and the Isabelle prover insisted on it before it would certify the theorem (see §4). When $v_i = v_j$, the two branches **both** fire: equality satisfies $v_i \le v_j$ **and** $v_j \le v_i$. This is the harmless overlap. Off the diagonal ($v_i \ne v_j$) exactly one branch fires and the colour is determined; on the diagonal both hold — the single square where the two colours of the board meet and the two branch-outcomes coincide. The lock is therefore stated as the two biconditionals above, each colour identical to its direction, with the diagonal as the place they touch.

3 The bishop stays on its colour

A single comparator locks parity to direction; the question is whether the lock survives **composition**, when comparators are chained into a network and branch atoms combine into the affine contexts of the discharge calculus. It does, and for a reason as simple as the base case.

Theorem 2 (Preservation)**.** *Let $c = (i, j)$ with $i \ne j$. Then*

$$O_c(v) \Rightarrow \big[(\mathit{apply\_comp}\ c\ v)_i \le (\mathit{apply\_comp}\ c\ v)_j \iff v_i \le v_j\big],$$

$$t_c(v) \Rightarrow \big[(\mathit{apply\_comp}\ c\ v)_i \le (\mathit{apply\_comp}\ c\ v)_j \iff v_j \le v_i\big].$$

In the even branch the comparator acts as the **identity** on the pair ($\mathrm{out}_i = v_i$, $\mathrm{out}_j = v_j$), so a forward output fact pulls back to a forward input fact: the colour is unchanged. In the odd branch the comparator acts as the **transposition** ($\mathrm{out}_i = v_j$, $\mathrm{out}_j = v_i$), so a forward output fact pulls back to the **reversed** input relation $v_j \le v_i$: again the colour is preserved, because the swap flips the parity and the direction **together**. A relabel through a transposition carries the colour with it. This is the bishop: whatever square it starts on, every legal move keeps it on that colour, because the move that would change the direction is exactly the move that changes the parity, and they cancel.

Corollary 1 (Two-colour decomposition)**.** *The affine branch-atom structure of a sorting-network derivation decomposes into two classes — **even/forward** and **odd/reversed** — welded to the two directions of the ordering relation. No comparator moves a coupling from one class to the other; each coupling is confined to its colour.*

4 Machine-checked

Theorems 1 and 2 are verified in Isabelle/HOL (`BranchAtoms.thy`, session `Bishop`, **Finished Bishop**, no `sorry`). They rest on the definitions

$$\texttt{branch\_0x}\; c\; v = (v_i \le v_j), \qquad \texttt{branch\_tx}\; c\; v = (v_j \le v_i),$$

which already carried the lock in their form, and on two resolution lemmas (`0x_resolves_min_max`, `tx_resolves_min_max`) establishing that each branch fixes the min/max to a definite wire (identity or transposition). The theorem names are `bishop_parity_direction_lock` and `bishop_preserved_by_comparat`

One methodological note, recorded because it is the honest history. The result was first sought **empirically** — by instrumenting the ledger and measuring whether accumulated couplings stayed on one colour. That instrument was mis-aimed six times over: it measured adjacent-but-wrong quantities (raw context parities, output-bit parities, closure classes), and its aggregate verdict oscillated. The correct move was to stop measuring and **read the mechanism**: the lock is forced at the source, in the two lines of the calculus that record a branch, and needs no measurement at all. And when the mechanism was handed to the prover, the prover **sharpened** it: a first, too-strong statement ($\neg$parity $\iff$ forward, ignoring the diagonal) failed to certify, and Isabelle produced exactly the equality case $v_i = v_j$ as the counterexample. The machine corrected the human's gloss. The final statement is truer for having been checked.

5 What it portends

We began with a small lemma and end with a claim about what sorting is.

A comparator's two branches are the two **parities** of a transposition: swap (odd) or not (even). Sorting is the resolution of a permutation to the identity, and the discharge calculus's affine layer is precisely the bookkeeping of **which transpositions were performed** — a record in the permutation group's $\mathrm{GF}(2)$/parity structure. The order the network builds is the $\le$ relation it certifies. These have long been treated as two structures: the **algebraic** (the group, the parity, the affine/XOR layer that resists monotone description) and the **order-theoretic** (the reachable set, the monotone BDD, the antichain of an order filter). Theorem 1 says they are not two structures. **The parity of a transposition and the direction it orders are the same bit.** The affine layer that Horn and median closure could not capture — the irreducible XOR — is exactly the ordering state's **directedness**, read as parity.

This is why, in the two-representation account of verification, parity never appears in a sorter's **output**. The sorted set is a chain; its reachable-set BDD is purely order-structured, with no XOR. One might have expected the parity information to demand its own place in the output representation. It does not, and the bishop explains why: the parity is fully discharged **into the ordering state's direction**. It lives in the derivation (which transpositions fired), not in the result (the order they produced), because in the result the two are welded into one. The duality between the order-BDD and the affine ledger is not a duality between two independent data structures; it is the duality between an order relation and the permutation parity that produced it — two faces of a single sorting computation, joined (as we found separately) by a Birkhoff correspondence on the order side and a novel parity–phantom correspondence on the affine side.

The bishop is the geometric image of this unity. Paint the board by ordering-direction: forward is one colour, reversed the other. A transposition's parity can only ever sit on its own colour, because the parity **is** the direction. There is no move that changes the order without changing the parity, and none that changes the parity without changing the order. On the diagonal — equality — the colours meet, and the distinction dissolves, because there is nothing to order and nothing to swap.

That two such different-looking accounts of sorting — the algebraist's permutation parity and the order theorist's monotone reachable set — turn out to be one object, welded at the level of a single comparator and preserved through every composition, is the kind of small structural fact that makes a subject feel, briefly, transparent. It was Heath's to see. It was a pleasure to prove.