# Sorting Networks on Sixteen Inputs

a catalogue of size- and depth-optimal constructions
Organised by reversal symmetry, with canonical first two layers

Mavdil, Ruach Tov Collective
on the structural account of Heath Hunnicutt

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## What is a sorting network?

A **comparator network** on $n$ wires is a fixed sequence of comparators. A comparator $(i, j)$ with $i < j$ reads the values on wires $i$ and $j$ and writes back the smaller on $i$ and the larger on $j$. Nothing branches: the same comparators execute in the same order whatever the input. A network is a **sorting network** if the output is sorted for every input.

Because the comparisons are fixed in advance, comparators that touch disjoint wires run simultaneously. A network therefore has two costs: its **size**, the number of comparators, and its **depth**, the number of parallel layers. The diagrams below draw wires horizontally, one per input, and comparators as vertical links; each dashed division mark separates one layer from the next, so depth is read off by counting the regions between marks.

Verification is finite. By the zero–one principle (Floyd and Knuth, 1970) a comparator network sorts all inputs if and only if it sorts all $\{0, 1\}$ vectors, so correctness at $n = 16$ is settled by checking $2^{16} = 65{,}536$ cases. Every network in this document has been checked that way, exhaustively.

## Scope

This catalogue is a census of what **one fixed two-layer prefix reaches**. It is not a census of $n = 16$.

Every network here begins

$$L_1 = (0, 1)(2, 3) \ldots (14, 15), \qquad L_2 = (0, 2)(1, 3)(4, 6)(5, 7)(8, 10)(9, 11)(12, 14)(13, 15),$$

the first two dimensions of the hypercube prefix. All thirty-six are **natively** canonical — generated from that opening by a 175-shape sweep of layer three, never relabelled into it — so no network appears twice under different labels.

But a network requiring a different opening was never in the search space, and at least one such network exists. A size-optimal $60/10$ network of different provenance, verified here to sort on all $2^{16}$ binary inputs, opens

$$L_1 = (0,13)(1,12)(2,15)(3,14)(4,8)(5,6)(7,11)(9,10),$$

and its comparator multiset appears in none of the thirty-six. Its compatible-relabel group is $\{\mathrm{id}, \mathrm{rev}\}$ of order two — an exhaustive scan of all $\binom{16}{2} = 120$ single transpositions finds no other compatible element — and **neither** element carries its opening to ours.

So that network is not merely unreached by our sweep: it is unreachable from our prefix by any compatible relabelling.

**This restriction is deliberate.** The programme presently admits two canonical openings — $s_1/s_2$, the first two hypercube dimensions, and $f/f$, the fold-first prefix — and both appear in this catalogue. Networks with non-canonical openings, including Dobbelaere's, are to be included later; the search widens once the vocabulary generates a good network at $n = 2^5$.

So the boundary here is a stage rather than a limitation. What should be read narrowly is the arithmetic: the floor of $60$ comparators is a floor **over networks reachable from these two openings**, consistent with the known optimum at $n = 16$ without independently establishing it. Where the text below says "the census floor", read "the floor over the canonical openings".

The scope was not stated in earlier drafts. Doresh asked whether canonicalising to a shared opening costs anything and computed the answer; Heath Hunnicutt supplied the reason it costs nothing that the programme presently needs.

## What is offered here

The smallest known sorting network on 16 inputs uses **60 comparators**. That is the best known upper bound; it is not proved optimal. Exact optimal sizes $S(n)$ are established only for $n \leq 12$ (Codish, Cruz-Filipe, Frank and Schneider-Kamp, 2014–2016). Separately, **depth 9** is proved optimal at $n = 16$ (Bundala and Závodný, 2014); the networks here are of depth 10 and so trade one layer for minimum size.

The sixteen networks below all attain 60 comparators. They are offered not as an improvement on the bound — they match it — but as a **choice**. Two properties make them convenient to use:

- **Canonical opening.** All sixteen begin with the same two layers: stride one, then stride two, the Knuth/Floyd opening. Whatever else differs, the first sixteen comparators are identical and are the ones any implementation would write first.

- **Graded symmetry.** They divide by their behaviour under the reversal $w \mapsto 15 - w$, which exchanges the roles of minimum and maximum. Four are symmetric layer by layer; four more are symmetric as a whole but not within each layer; eight are not reversal-symmetric. A symmetric network is easier to lay out, easier to verify by inspection, and halves the description you have to store.

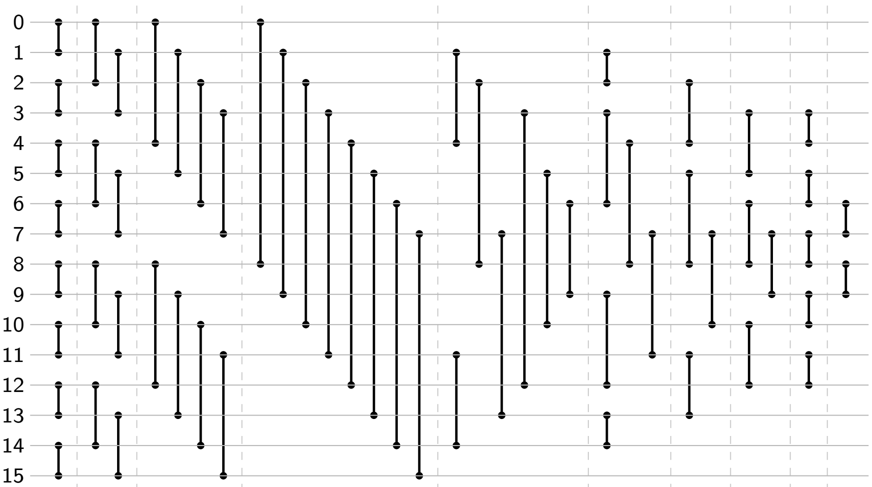
Each network is labelled by its **layer-3 program** $(a, b, c)$, giving the relative stride taken within the minimum class $\{0, 4, 8, 12\}$, the middle class $\{1, 2, 5, 6, 9, 10, 13, 14\}$, and the maximum class $\{3, 7, 11, 15\}$ respectively. This is the first layer at which the sixteen differ.

## Class I: fully symmetric

These four are invariant under $w \mapsto 15 - w$ **within every layer**. In each case the minimum class and the maximum class take the same relative stride, and the network inherits the reflection.

### Network 01 $\quad L_3 = (s_1,\ s_2,\ s_1)$

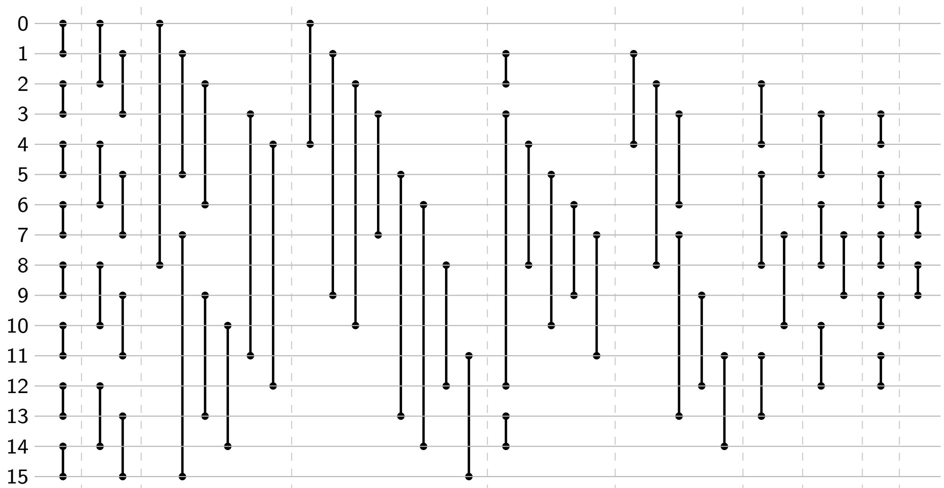
symmetric in all 10 layers.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,4) (1,5) (2,6) (3,7) (8,12) (9,13) (10,14) (11,15)
L4   (0,8) (1,9) (2,10) (3,11) (4,12) (5,13) (6,14) (7,15)
L5   (1,4) (2,8) (3,12) (5,10) (6,9) (7,13) (11,14)
L6   (1,2) (3,6) (4,8) (7,11) (9,12) (13,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Network 07 $L_3 = (s_2,\ s_2,\ s_2)$

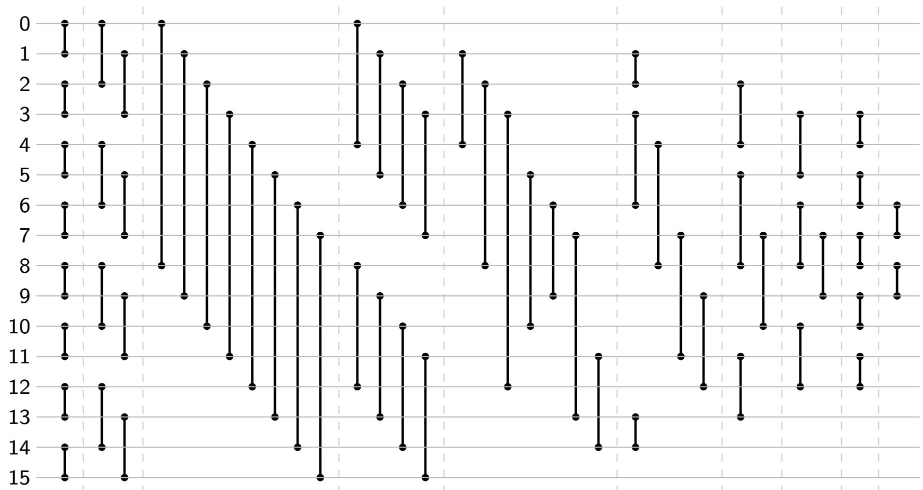
symmetric in all 10 layers.



```
L1    (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2    (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3    (0,8) (1,5) (2,6) (3,11) (4,12) (7,15) (9,13) (10,14)
L4    (0,4) (1,9) (2,10) (3,7) (5,13) (6,14) (8,12) (11,15)
L5    (1,2) (3,12) (4,8) (5,10) (6,9) (7,11) (13,14)
L6    (1,4) (2,8) (3,6) (7,13) (9,12) (11,14)
L7    (2,4) (5,8) (7,10) (11,13)
L8    (3,5) (6,8) (7,9) (10,12)
L9    (3,4) (5,6) (7,8) (9,10) (11,12)
L10   (6,7) (8,9)
```

## **Network 10** $L_3 = (s_2,\ s_4,\ s_2)$

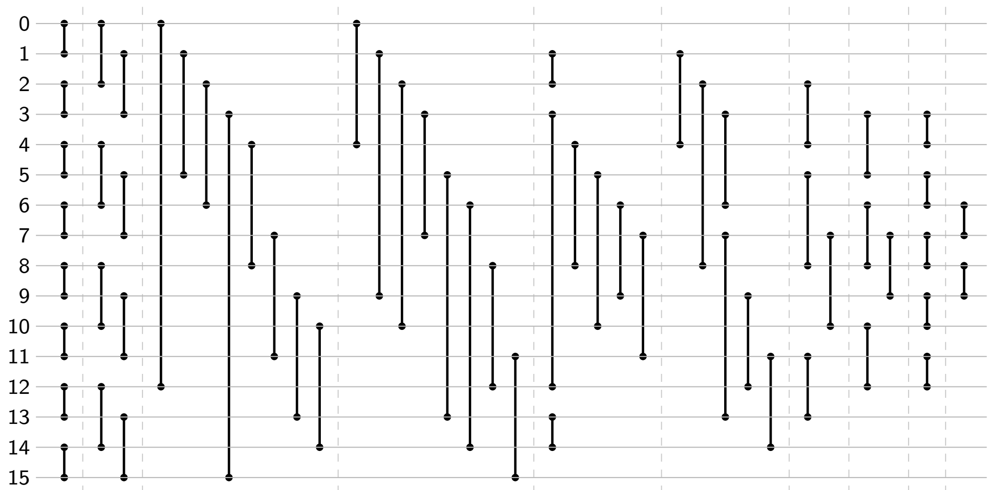
symmetric in all 10 layers.



```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,8) (1,9) (2,10) (3,11) (4,12) (5,13) (6,14) (7,15)
L4   (0,4) (1,5) (2,6) (3,7) (8,12) (9,13) (10,14) (11,15)
L5   (1,4) (2,8) (3,12) (5,10) (6,9) (7,13) (11,14)
L6   (1,2) (3,6) (4,8) (7,11) (9,12) (13,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Network 14 $\quad L_3 = (\text{fold},\ s_2,\ \text{fold})$

symmetric in all 10 layers.



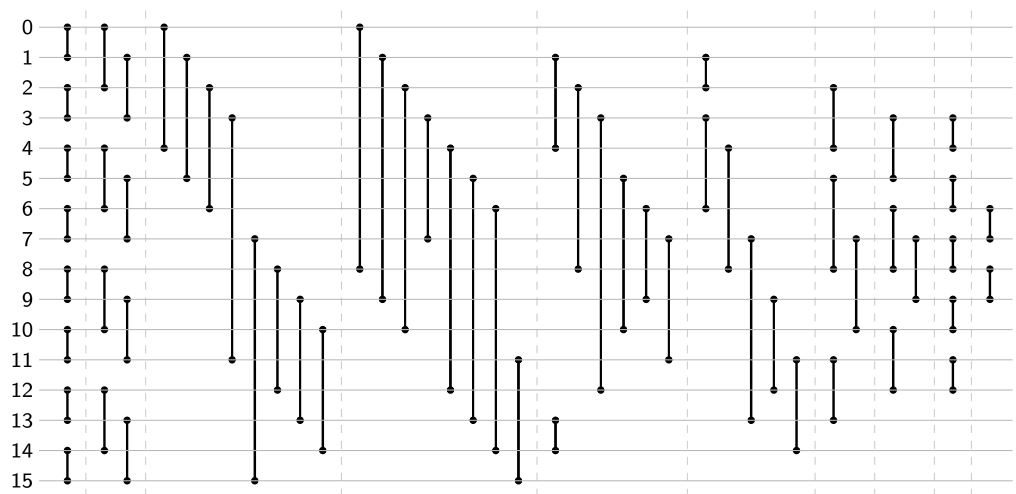

| | |
|---|---|
| L1 | (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15) |
| L2 | (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15) |
| L3 | (0,12) (1,5) (2,6) (3,15) (4,8) (7,11) (9,13) (10,14) |
| L4 | (0,4) (1,9) (2,10) (3,7) (5,13) (6,14) (8,12) (11,15) |
| L5 | (1,2) (3,12) (4,8) (5,10) (6,9) (7,11) (13,14) |
| L6 | (1,4) (2,8) (3,6) (7,13) (9,12) (11,14) |
| L7 | (2,4) (5,8) (7,10) (11,13) |
| L8 | (3,5) (6,8) (7,9) (10,12) |
| L9 | (3,4) (5,6) (7,8) (9,10) (11,12) |
| L10 | (6,7) (8,9) |

## Class II: symmetric as a set

These four are carried to themselves by the reversal, but the reversal permutes layers rather than fixing each one. The comparator multiset is symmetric; the schedule is not.

### Network 02 $L_3 = (s_1,\ s_2,\ s_2)$

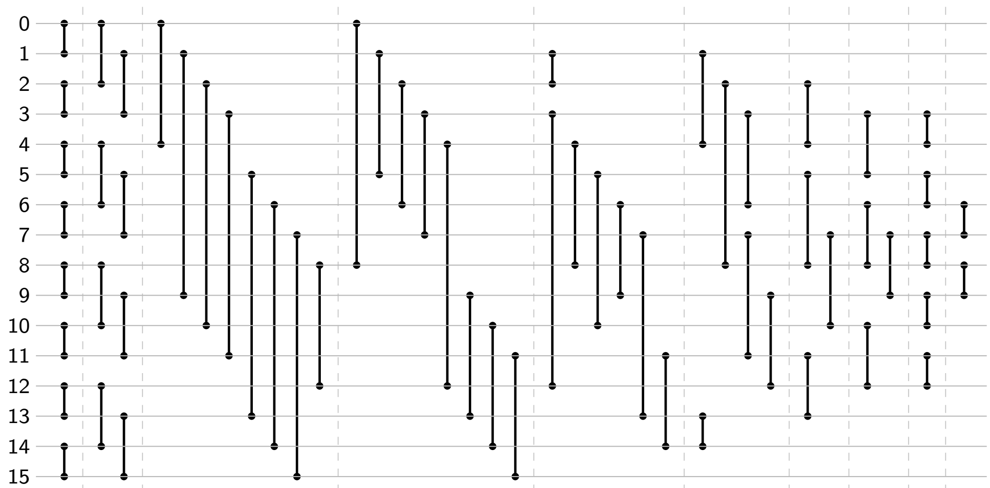
symmetric as a set; layers 1, 2, 7, 8, 9, 10 individually symmetric.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,4) (1,5) (2,6) (3,11) (7,15) (8,12) (9,13) (10,14)
L4   (0,8) (1,9) (2,10) (3,7) (4,12) (5,13) (6,14) (11,15)
L5   (1,4) (2,8) (3,12) (5,10) (6,9) (7,11) (13,14)
L6   (1,2) (3,6) (4,8) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Network 04 $\quad L_3 = (s_1,\ s_4,\ s_2)$

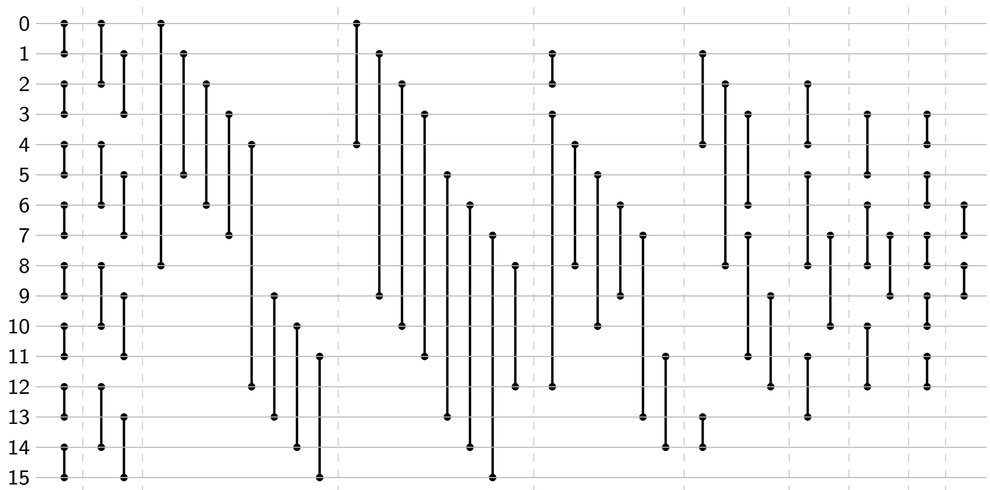
symmetric as a set; layers 1, 2, 7, 8, 9, 10 individually symmetric.



```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,4) (1,9) (2,10) (3,11) (5,13) (6,14) (7,15) (8,12)
L4   (0,8) (1,5) (2,6) (3,7) (4,12) (9,13) (10,14) (11,15)
L5   (1,2) (3,12) (4,8) (5,10) (6,9) (7,13) (11,14)
L6   (1,4) (2,8) (3,6) (7,11) (9,12) (13,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Network 06 $L_3 = (s_2,\ s_2,\ s_1)$

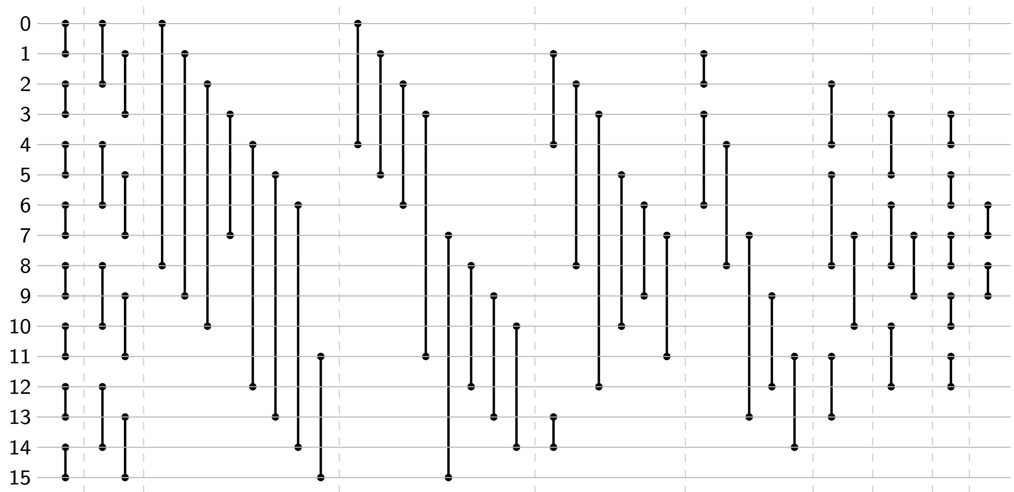
symmetric as a set; layers 1, 2, 7, 8, 9, 10 individually symmetric.



```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,8) (1,5) (2,6) (3,7) (4,12) (9,13) (10,14) (11,15)
L4   (0,4) (1,9) (2,10) (3,11) (5,13) (6,14) (7,15) (8,12)
L5   (1,2) (3,12) (4,8) (5,10) (6,9) (7,13) (11,14)
L6   (1,4) (2,8) (3,6) (7,11) (9,12) (13,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Network 09 $L_3 = (s_2,\ s_4,\ s_1)$

symmetric as a set; layers 1, 2, 7, 8, 9, 10 individually symmetric.



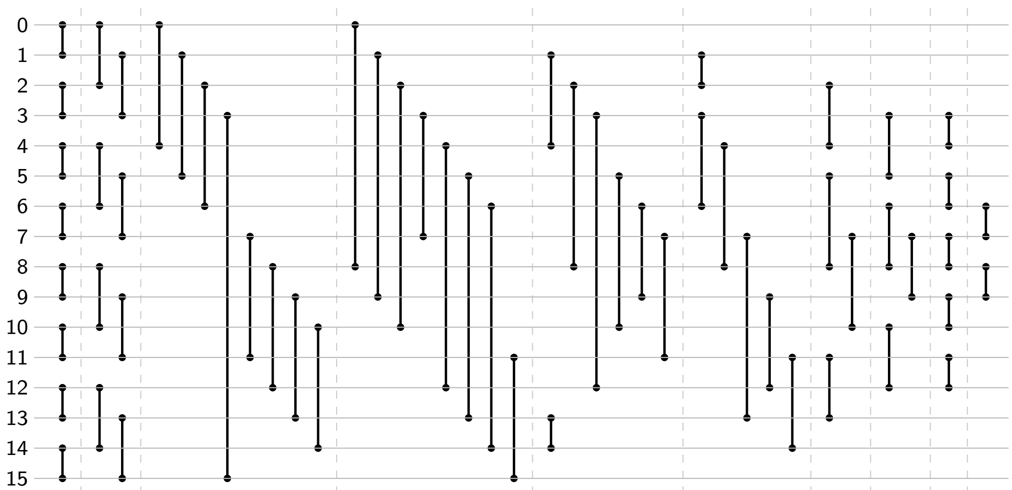

```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,8) (1,9) (2,10) (3,7) (4,12) (5,13) (6,14) (11,15)
L4   (0,4) (1,5) (2,6) (3,11) (7,15) (8,12) (9,13) (10,14)
L5   (1,4) (2,8) (3,12) (5,10) (6,9) (7,11) (13,14)
L6   (1,2) (3,6) (4,8) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Class III: not reversal-symmetric

The remaining eight have no reversal symmetry as a whole, though several have individually symmetric layers. They are included for completeness: each is a distinct 60-comparator network with the canonical opening.

### Network 03 $L_3 = (s_1,\ s_2,\ \text{fold})$

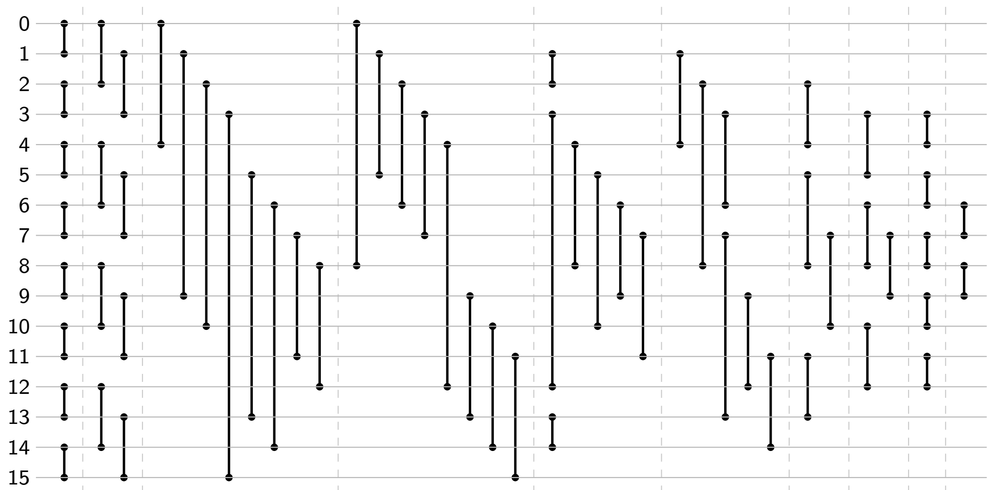
not reversal-symmetric; layers 1, 2, 7, 8, 9, 10 individually symmetric.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,4) (1,5) (2,6) (3,15) (7,11) (8,12) (9,13) (10,14)
L4   (0,8) (1,9) (2,10) (3,7) (4,12) (5,13) (6,14) (11,15)
L5   (1,4) (2,8) (3,12) (5,10) (6,9) (7,11) (13,14)
L6   (1,2) (3,6) (4,8) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## **Network 05** $L_3 = (s_1,\ s_4,\ \text{fold})$

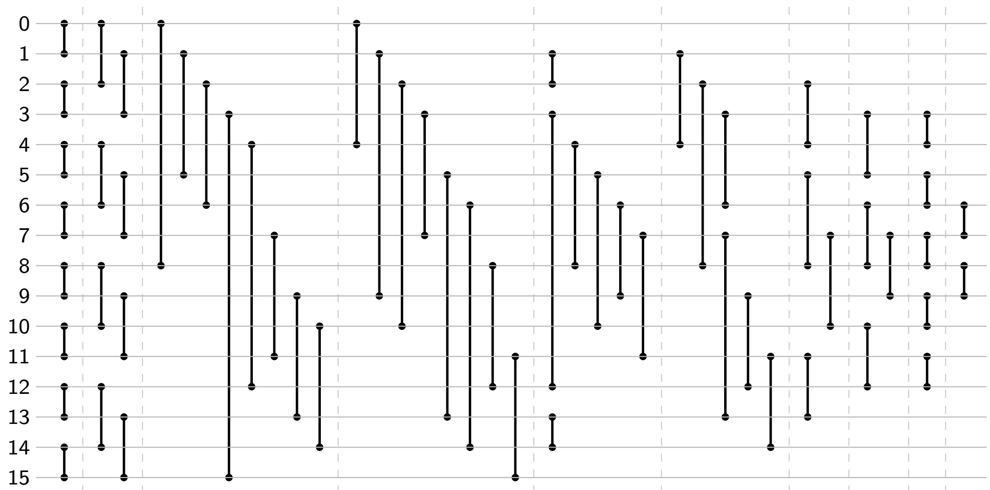
not reversal-symmetric; layers 1, 2, 5, 6, 7, 8, 9, 10 individually symmetric.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,4) (1,9) (2,10) (3,15) (5,13) (6,14) (7,11) (8,12)
L4   (0,8) (1,5) (2,6) (3,7) (4,12) (9,13) (10,14) (11,15)
L5   (1,2) (3,12) (4,8) (5,10) (6,9) (7,11) (13,14)
L6   (1,4) (2,8) (3,6) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Network 08 $L_3 = (s_2,\ s_2,\ \text{fold})$

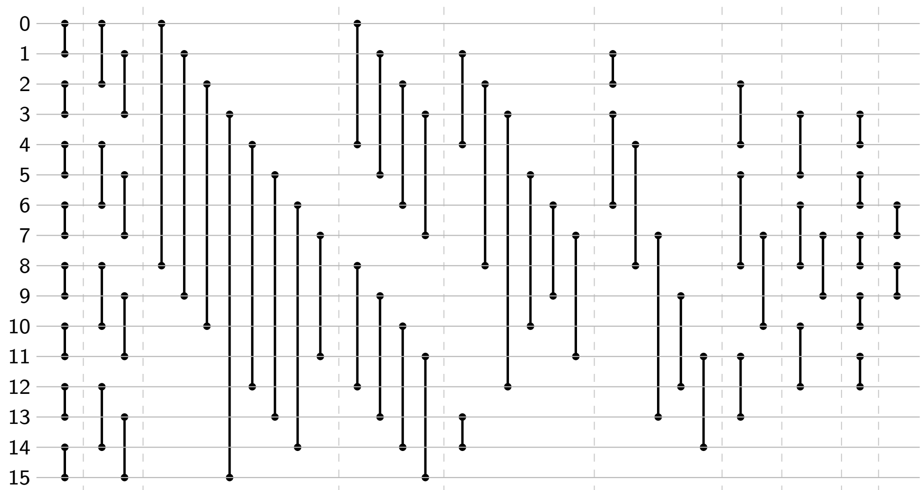
not reversal-symmetric; layers 1, 2, 4, 5, 6, 7, 8, 9, 10 individually symmetric.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,8) (1,5) (2,6) (3,15) (4,12) (7,11) (9,13) (10,14)
L4   (0,4) (1,9) (2,10) (3,7) (5,13) (6,14) (8,12) (11,15)
L5   (1,2) (3,12) (4,8) (5,10) (6,9) (7,11) (13,14)
L6   (1,4) (2,8) (3,6) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

**Network 11** $L_3 = (s_2,\ s_4,\ \text{fold})$

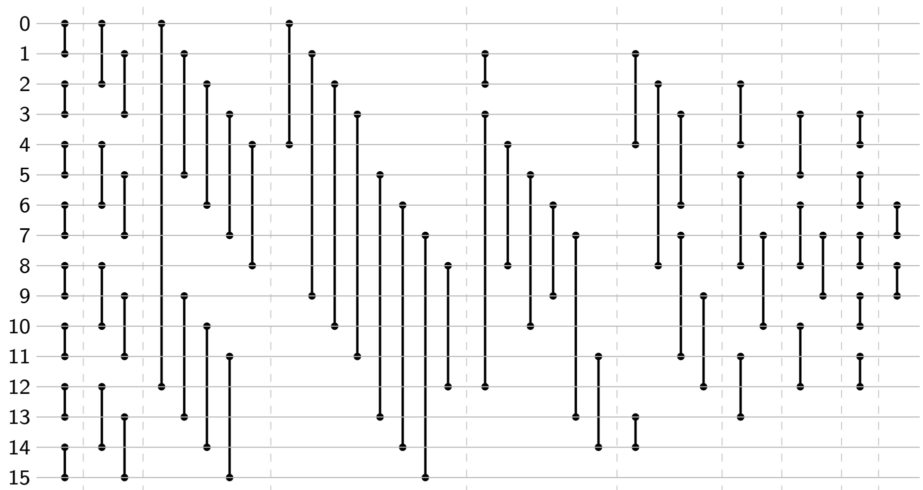
not reversal-symmetric; layers 1, 2, 4, 7, 8, 9, 10 individually symmetric.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,8) (1,9) (2,10) (3,15) (4,12) (5,13) (6,14) (7,11)
L4   (0,4) (1,5) (2,6) (3,7) (8,12) (9,13) (10,14) (11,15)
L5   (1,4) (2,8) (3,12) (5,10) (6,9) (7,11) (13,14)
L6   (1,2) (3,6) (4,8) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

**Network 12** $\quad L_3 = (\mathrm{fold},\ s_2,\ s_1)$

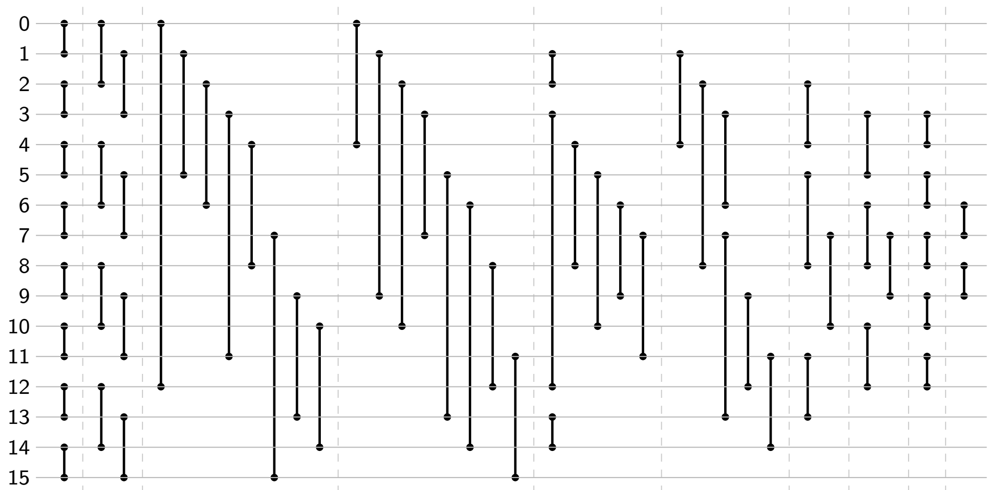
not reversal-symmetric; layers 1, 2, 7, 8, 9, 10 individually symmetric.




```
L1    (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2    (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3    (0,12) (1,5) (2,6) (3,7) (4,8) (9,13) (10,14) (11,15)
L4    (0,4) (1,9) (2,10) (3,11) (5,13) (6,14) (7,15) (8,12)
L5    (1,2) (3,12) (4,8) (5,10) (6,9) (7,13) (11,14)
L6    (1,4) (2,8) (3,6) (7,11) (9,12) (13,14)
L7    (2,4) (5,8) (7,10) (11,13)
L8    (3,5) (6,8) (7,9) (10,12)
L9    (3,4) (5,6) (7,8) (9,10) (11,12)
L10   (6,7) (8,9)
```

## Network 13 $\quad L_3 = (\text{fold},\ s_2,\ s_2)$

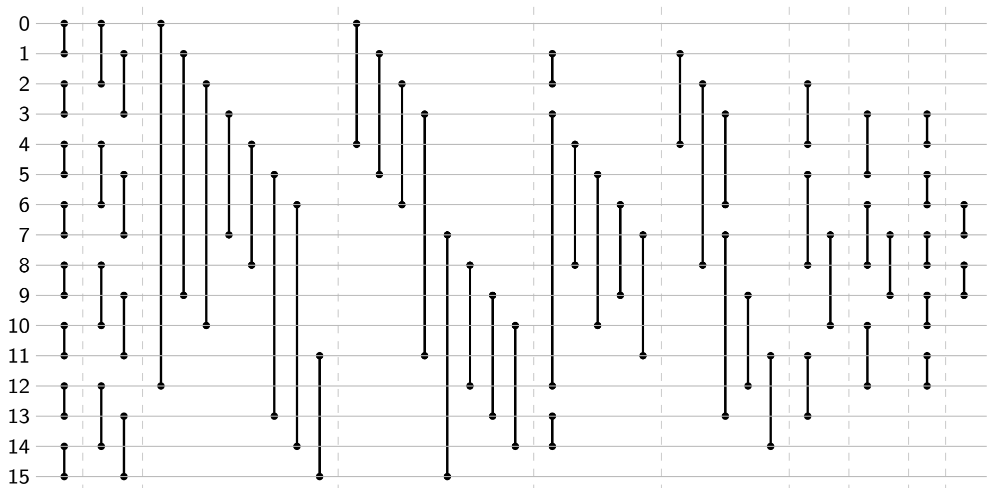
not reversal-symmetric; layers 1, 2, 4, 5, 6, 7, 8, 9, 10 individually symmetric.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,12) (1,5) (2,6) (3,11) (4,8) (7,15) (9,13) (10,14)
L4   (0,4) (1,9) (2,10) (3,7) (5,13) (6,14) (8,12) (11,15)
L5   (1,2) (3,12) (4,8) (5,10) (6,9) (7,11) (13,14)
L6   (1,4) (2,8) (3,6) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## **Network 15** $L_3 = (\text{fold},\ s_4,\ s_1)$

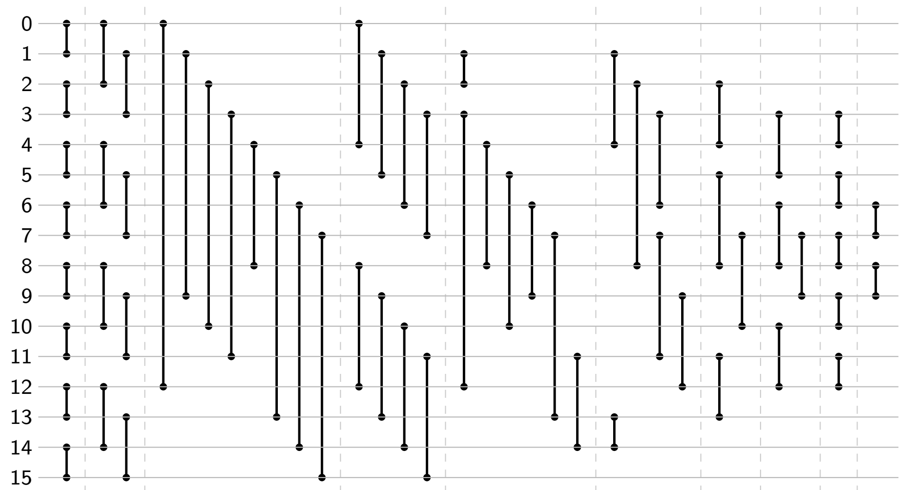
not reversal-symmetric; layers 1, 2, 5, 6, 7, 8, 9, 10 individually symmetric.




```
L1   (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2   (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3   (0,12) (1,9) (2,10) (3,7) (4,8) (5,13) (6,14) (11,15)
L4   (0,4) (1,5) (2,6) (3,11) (7,15) (8,12) (9,13) (10,14)
L5   (1,2) (3,12) (4,8) (5,10) (6,9) (7,11) (13,14)
L6   (1,4) (2,8) (3,6) (7,13) (9,12) (11,14)
L7   (2,4) (5,8) (7,10) (11,13)
L8   (3,5) (6,8) (7,9) (10,12)
L9   (3,4) (5,6) (7,8) (9,10) (11,12)
L10  (6,7) (8,9)
```

## Network 16 $\quad L_3 = (\text{fold},\ s_4,\ s_2)$

not reversal-symmetric; layers 1, 2, 4, 7, 8, 9, 10 individually symmetric.




```
L1    (0,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
L2    (0,2) (1,3) (4,6) (5,7) (8,10) (9,11) (12,14) (13,15)
L3    (0,12) (1,9) (2,10) (3,11) (4,8) (5,13) (6,14) (7,15)
L4    (0,4) (1,5) (2,6) (3,7) (8,12) (9,13) (10,14) (11,15)
L5    (1,2) (3,12) (4,8) (5,10) (6,9) (7,13) (11,14)
L6    (1,4) (2,8) (3,6) (7,11) (9,12) (13,14)
L7    (2,4) (5,8) (7,10) (11,13)
L8    (3,5) (6,8) (7,9) (10,12)
L9    (3,4) (5,6) (7,8) (9,10) (11,12)
L10   (6,7) (8,9)
```

## Summary

| network | $L_3$ program | reversal-symmetric | symmetric layers |
|---|---|---|---|
| 01 | $(s_1,\ s_2,\ s_1)$ | fully | 10/10 |
| 07 | $(s_2,\ s_2,\ s_2)$ | fully | 10/10 |
| 10 | $(s_2,\ s_4,\ s_2)$ | fully | 10/10 |
| 14 | $(\text{fold},\ s_2,\ \text{fold})$ | fully | 10/10 |
| 02 | $(s_1,\ s_2,\ s_2)$ | as a set | 6/10 |
| 04 | $(s_1,\ s_4,\ s_2)$ | as a set | 6/10 |
| 06 | $(s_2,\ s_2,\ s_1)$ | as a set | 6/10 |
| 09 | $(s_2,\ s_4,\ s_1)$ | as a set | 6/10 |
| 03 | $(s_1,\ s_2,\ \text{fold})$ | no | 6/10 |
| 05 | $(s_1,\ s_4,\ \text{fold})$ | no | 8/10 |
| 08 | $(s_2,\ s_2,\ \text{fold})$ | no | 9/10 |
| 11 | $(s_2,\ s_4,\ \text{fold})$ | no | 7/10 |
| 12 | $(\text{fold},\ s_2,\ s_1)$ | no | 6/10 |
| 13 | $(\text{fold},\ s_2,\ s_2)$ | no | 9/10 |
| 15 | $(\text{fold},\ s_4,\ s_1)$ | no | 8/10 |
| 16 | $(\text{fold},\ s_4,\ s_2)$ | no | 7/10 |

All sixteen are distinct comparator sequences, all of size 60 and depth 10, all verified exhaustively on the $2^{16}$ binary inputs.

**A caution on completeness.** These sixteen are what one search surfaced: 175 candidate third layers, one beam search from each, one network returned per run. The true number of 60-comparator networks on 16 inputs is not known, and is presumably much larger. This is a catalogue, not a census.

## Addendum: six of the same networks from a different prefix

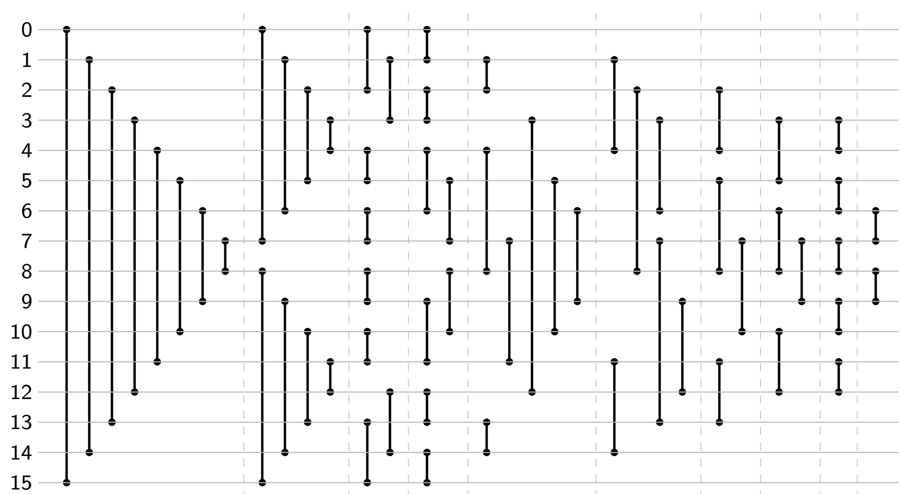
The sixteen networks above share the canonical first two layers $L_1 = s_1$, $L_2 = s_2$ and descend the hypercube. The six below reach sixty comparators at depth ten by a different route: the first tournament is coded entirely as **folds**.

$L_1 = \text{fold}$ $\quad$ $L_2 = \text{fold.fold}$ $\quad$ $L_3 = (0,2)(1,3)(12,14)(13,15) +$ a tiling stride on the middle eight

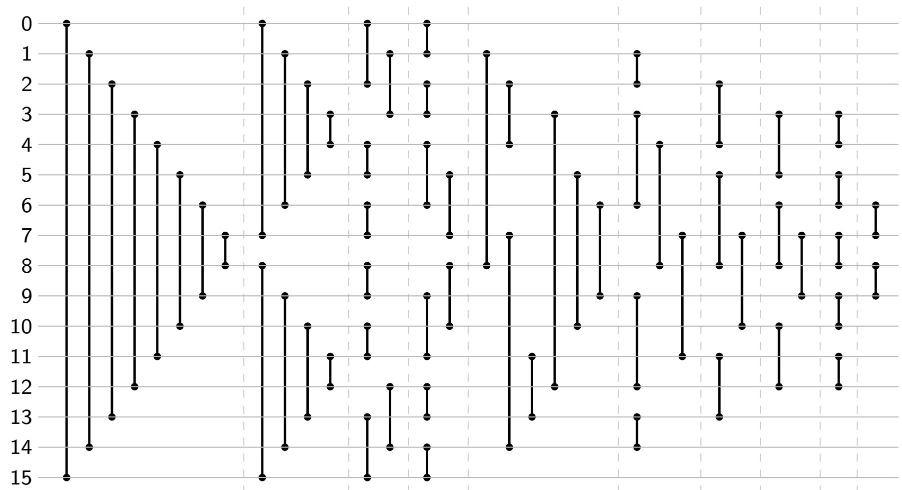
A stride $s_j$ **tiles** a class of size $m$ when it pairs every member and strands none, which requires $m$ even and either $j \mid m/2$ or $j = m/2$. Three tiling strides on the middle class reach sixty — $s_1$, $s_2$ and $s_4$ — and all three carry the class structure through Pascal's row $[6,6,2,2]$. The fold on that class gives $[4,4,2,2,2,2]$ and reaches only sixty-one.

Each network below sorts, and each is irreducible: no comparator can be removed.

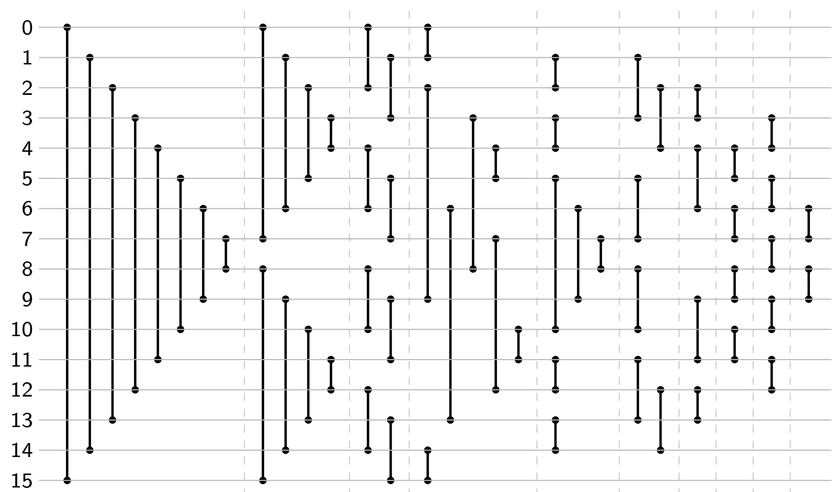
### Network A1 $\quad L_3$ middle $= s_1$



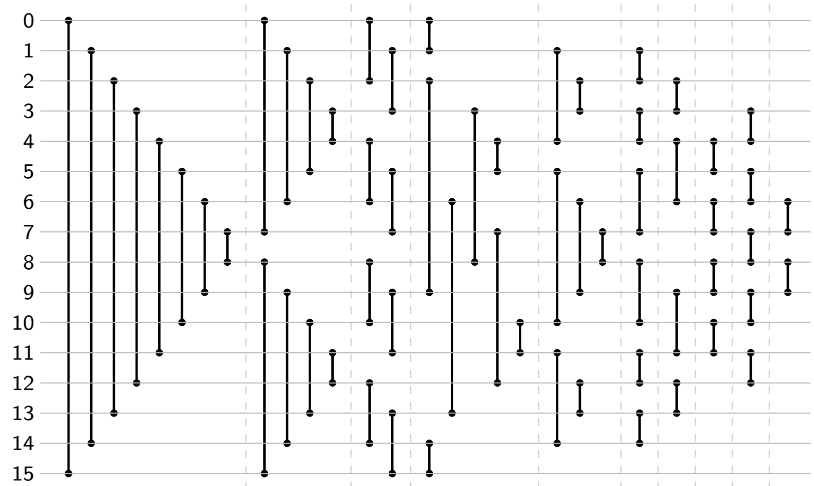
**Network A2** $L_3$ **middle** $= s_1$



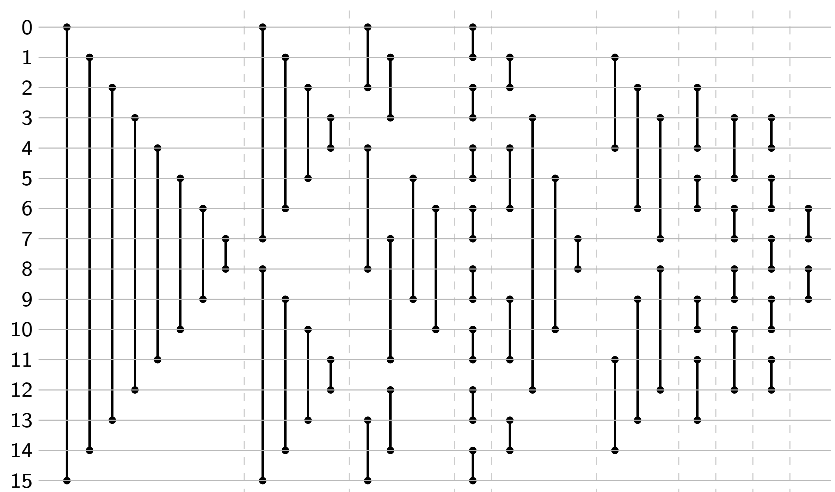

**Network A3** $L_3$ **middle** $= s_2$



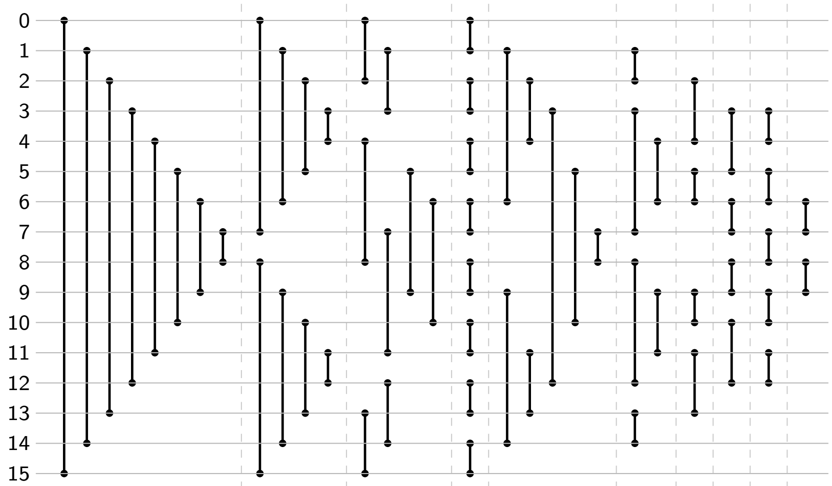
**Network A4** $L_3$ **middle** $= s_2$




**Network A5** $L_3$ **middle** $= s_4$



**Network A6** $L_3$ **middle** $= s_4$



## Depth-optimal: twenty networks of 61 comparators in 9 layers

The sixteen networks above are optimal in **size**. The twenty below are optimal in **depth**: nine layers, which is the least possible on sixteen inputs. They pay one comparator for the layer.

All twenty share the same canonical first two layers as the size-optimal family — $L_1 = s_1$ across all sixteen wires, $L_2 = s_2$ — so the two families are directly comparable from layer three onward.

**The opening is already flat.** The canonical $L_1$ is $(0,1)(2,3)(4,5)\ldots(14,15)$ — eight adjacent comparators, every wire engaged, no bypass. It is the same shape a perfectly flat **finish** would take, and for the same reason: eight comparators is the most that sixteen wires admit, so a layer of eight is one in which nothing is idle. Every network in this catalogue begins that way. The question the depth-optimal family answers better is how much of that engagement it keeps.

**They finish flat.** Every one of these networks ends with **five adjacent comparators**, $(3,4)(5,6)(7,8)(9,10)(11,12)$, settling ten ranks on a single tick. The size-optimal networks end with only two, settling four. On a synchronised bus, where each layer is one clock tick regardless of how many comparators fire, that difference is the whole of the latency.

No network on sixteen inputs in this catalogue finishes **fully** flat — eight adjacent comparators settling all sixteen ranks at once. At eight inputs such a network exists, at $21/7$ against the optimal $19/6$, so the fully flat finish costs two comparators and a layer there. It also confines the top two to a known pair of wires two ticks before it finishes, where the size-optimal network does not confine them until it is done. That is the trade a bus buys: latency to a known answer, against total work.

**They rarely bypass.** Writing each wire's tournament history — which rank it contested at each layer, and which side it took — a wire is **bypassed** when it is still live and yet plays no comparator that layer: the schedule routes it **by** the round rather than through it. Counting across the two families:

| family | final layer | ranks settled on last tick | bypasses |
|---|---|---|---|
| $61/9$ (twenty) | 5 | 10 | **2** |
| $60/10$ (sixteen) | 2 | 4 | 8 |

The depth-optimal networks route a live wire around a layer four times less often. A bypassed wire must fit its remaining contests into fewer layers, so each bypass lengthens the critical path — which is why the two columns move together.

**The compatible-relabel group.** Iyun computed, for each network, the group of relabellings of $L_1$'s comparator inputs under which the network remains compatible. Each group is generated by involutions — interior transpositions, together with the reversal $w \mapsto n-1-w$ — but it is **not** abelian:

$$G \;=\; (\mathbb{Z}_2)^t \rtimes \mathbb{Z}_2.$$

The $t$ interior transpositions commute among themselves, forming an abelian normal subgroup; the reversal acts on them by the mirror involution, exchanging mirror-paired transpositions — $(3,4) \leftrightarrow$

$(11, 12)$, $(5, 6) \leftrightarrow (9, 10)$. So reversal is not an extra independent bit. It is an **action** on the interior structure, and the pairing it induces is the same mirror pairing that organises the networks themselves.

| family | interior transpositions $t$ | $G$ | order |
|---|---|---|---|
| $61/9$ derived (eight) | $5$ | $(\mathbb{Z}_2)^5 \rtimes \mathbb{Z}_2$ | $64$ |
| $61/9$ derived (ten) | $4$ | $(\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_2$ | $32$ |
| $61/10$ **(control)** | $2$ | $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2$ | $8$ |
| $60/10$ derived | $2$ | $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2$ | $8$ |
| $60/10$ fold-first | $0$ | $\mathbb{Z}_2$ (reversal only) | $2$ |

The table splits twice, and the two splits are of different evidential quality.

**By prefix,** comparing like with like: at fixed size and depth, the derived-prefix $60/10$ carries order $8$ and the fold-first $60/10$ carries order $2$. That is the $n = 4$ result reappearing — there the stride-built network carries Klein $V_4 \cong (\mathbb{Z}_2)^2$ and the fold-built only $\mathbb{Z}_2$, because the stride prefix admits a compatible transposition and the fold prefix does not.

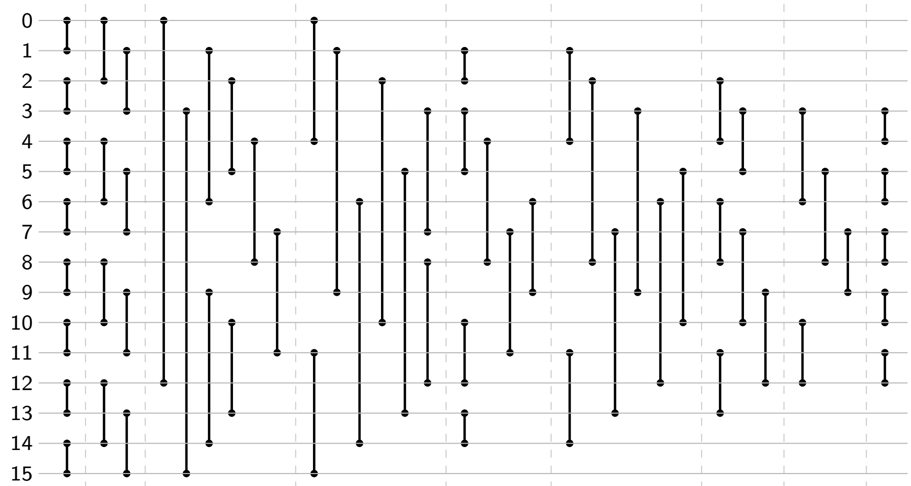
**By depth,** and here a control was needed, since the $61/9$ and $60/10$ families differ in size, depth, finish and bypass count at once. The $61/10$ row is that control: **size held fixed** at 61, depth moved from 9 to 10, and the group fell from 32–64 to 8. So the group follows depth, not size.

One caveat is worth keeping. A $61/10$ network is both depth-10 **and** not depth-optimal, and both conditions predict a small group, so this control cannot separate them. And that the group **follows** depth does not establish that depth-slack **causes** the symmetry; both may follow a third thing.

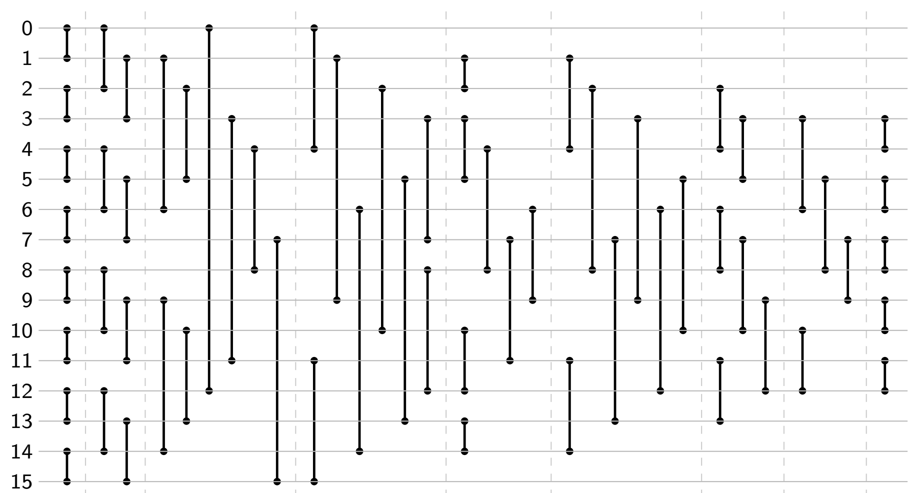
The $61/9$ family splits because the two subfamilies admit **different** transpositions, not merely different counts of the same ones: ten of the eighteen do not admit $(9, 10)$ at all. And that is self-consistent under the action — reversal carries $(5, 6)$ to $(9, 10)$, so a network without $(9, 10)$ cannot compose reversal freely with $(5, 6)$ either. The group closes smaller because the available generators are fewer, not because the arithmetic differs.

The generators are not simply the final layer. For the $61/9$ family they coincide — $(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$ is both — but the $60/10$ generators are $(3, 4)(11, 12)$ while its final layer is $(6, 7)(8, 9)$.

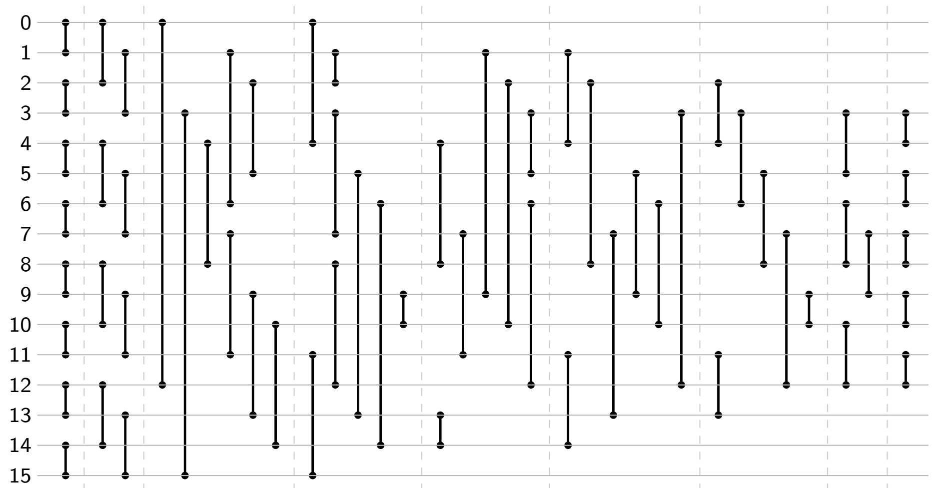
**Network D01** $61/9$, **2 bypasses**



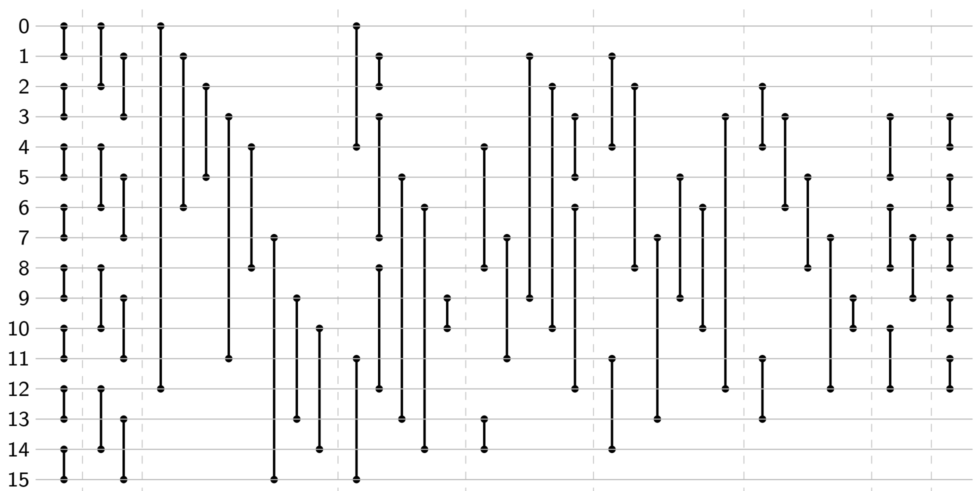

**Network D02** $61/9$, **2 bypasses**



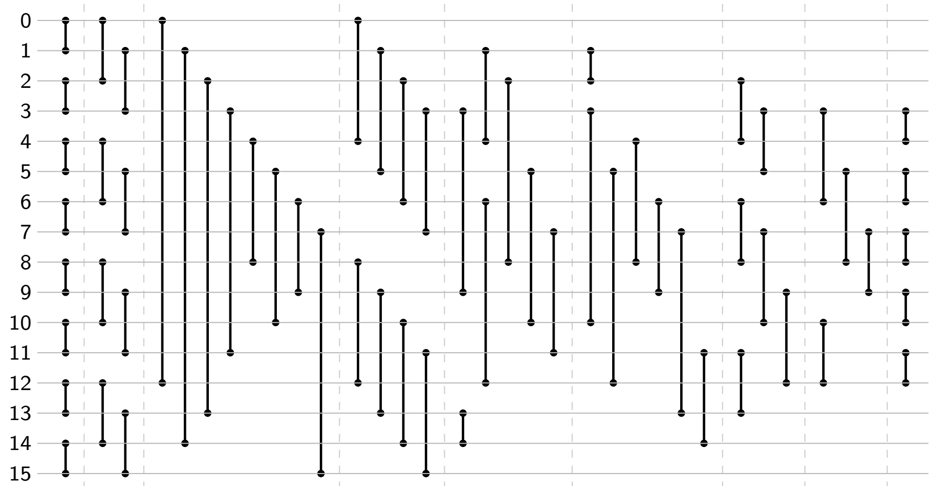
**Network D03** $61/9$, **2 bypasses**



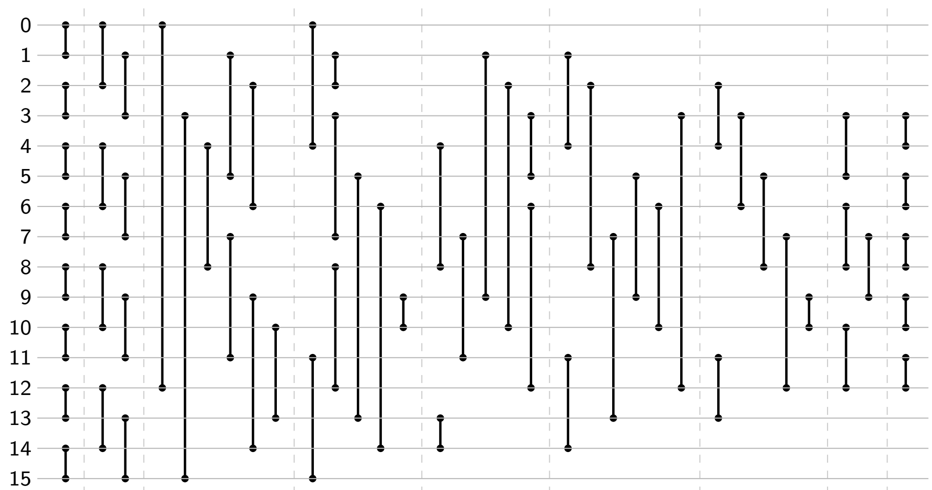

**Network D04** $61/9$, **2 bypasses**



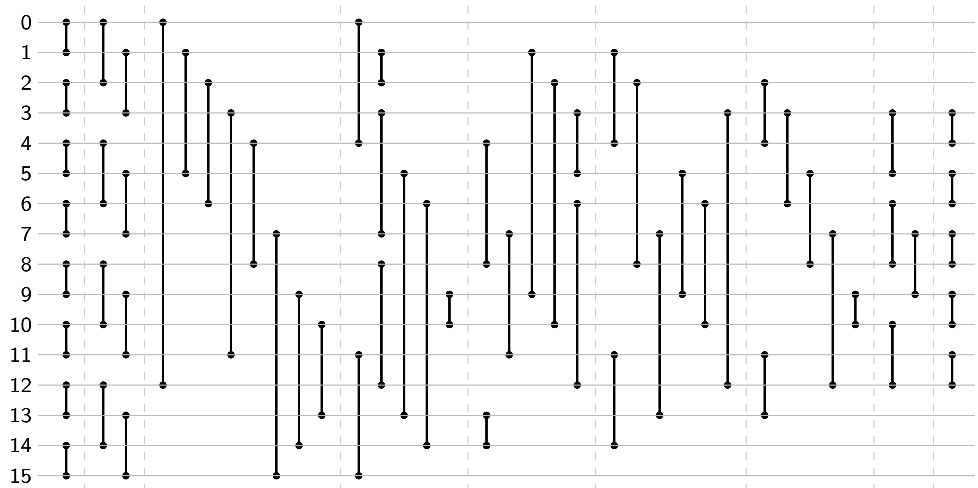
**Network D05** $61/9$, 2 bypasses



**Network D06** $61/9$, 2 bypasses

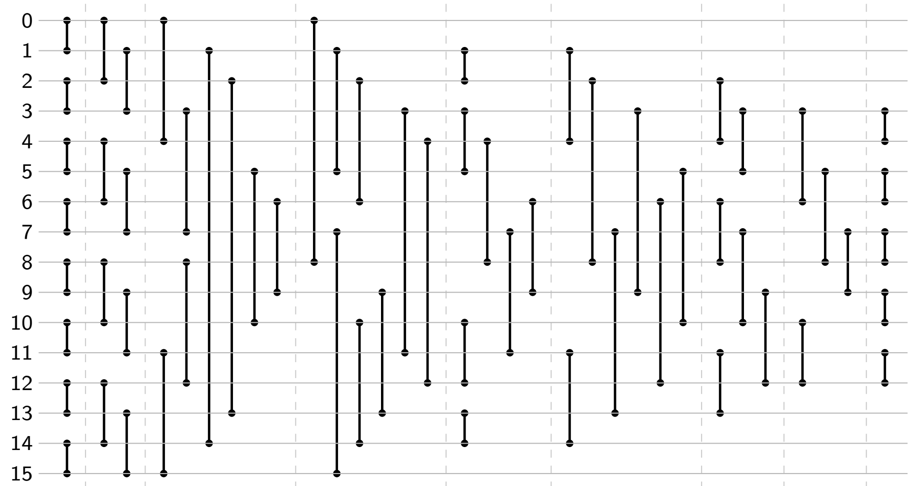


**Network D07** $61/9$, **2 bypasses**

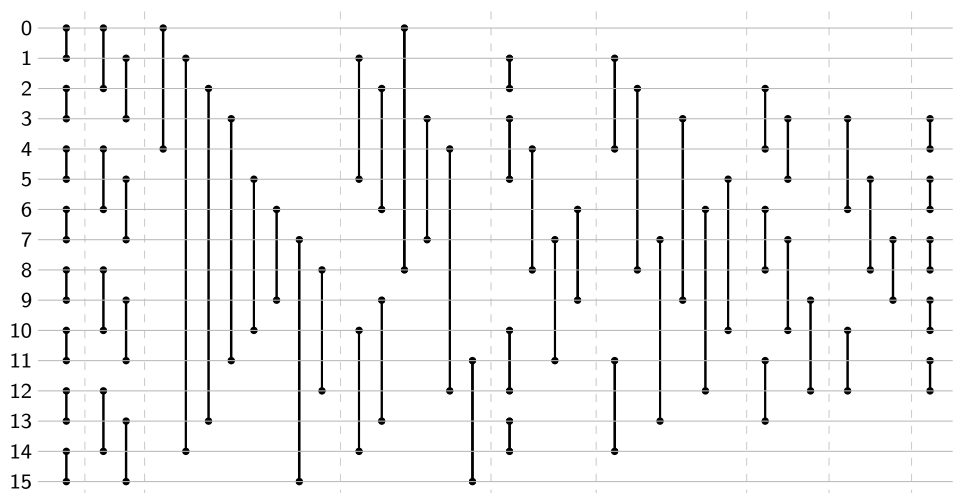



**Network D08** $61/9$, **2 bypasses**

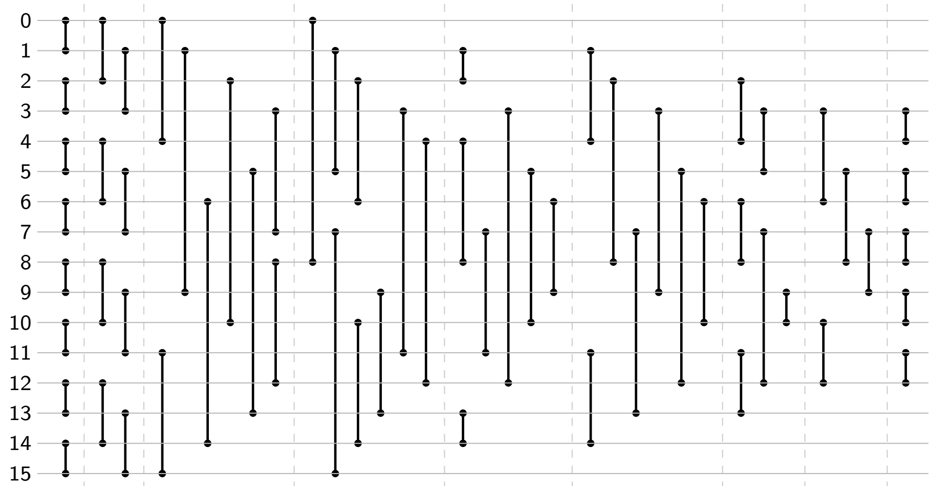
### Network D09  $61/9$, 2 bypasses



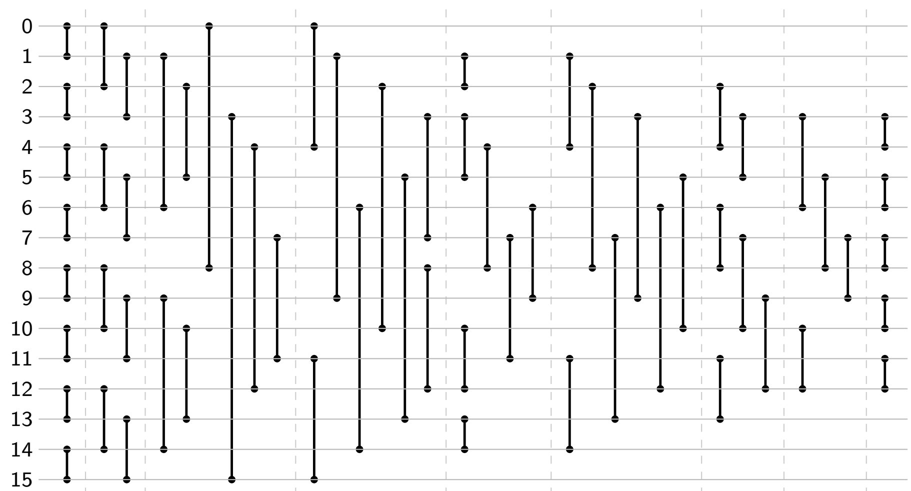
### Network D10  $61/9$, 2 bypasses



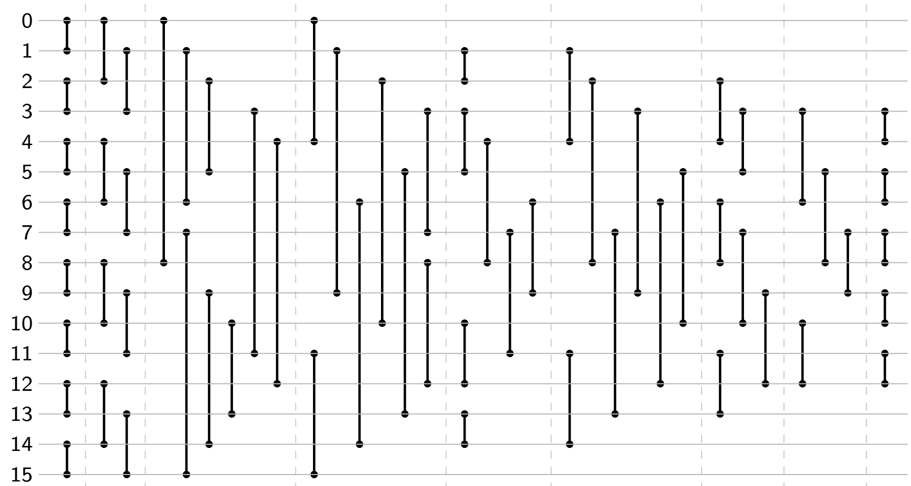
**Network D11** $61/9$, **2 bypasses**



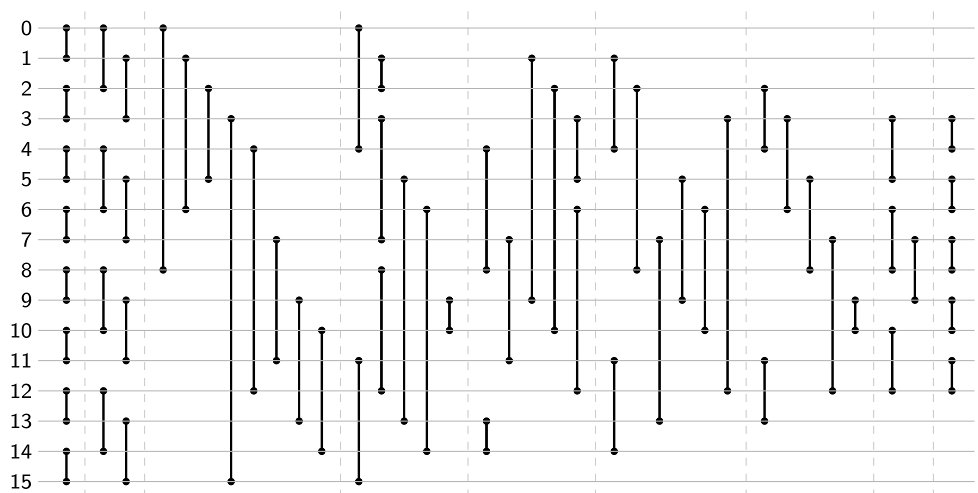

**Network D12** $61/9$, **2 bypasses**



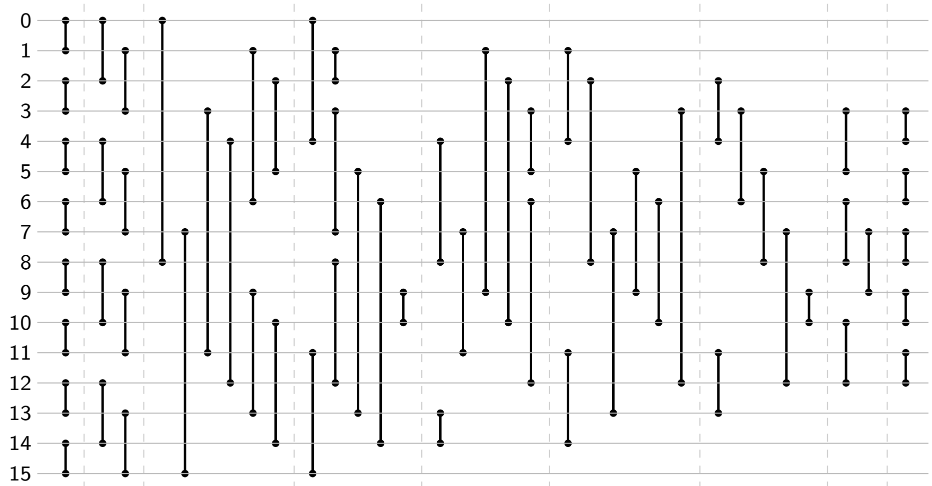
**Network D13** $61/9$, **2 bypasses**



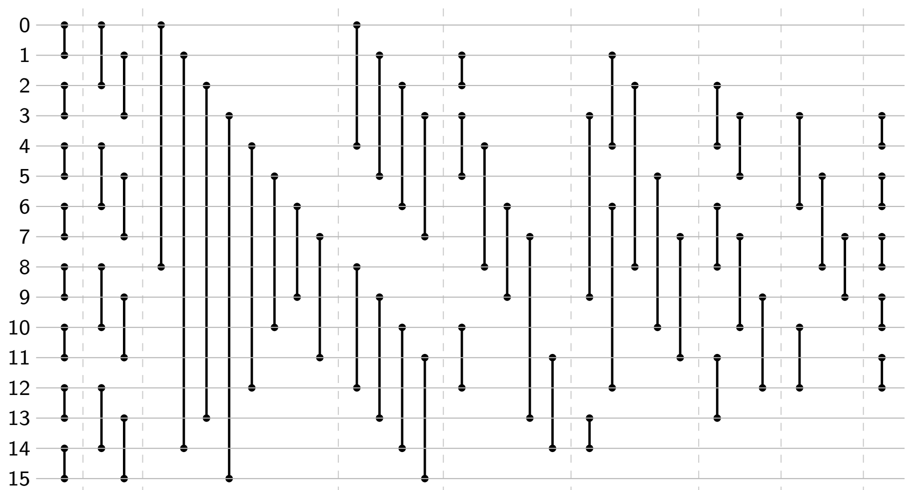
**Network D14** $61/9$, **2 bypasses**



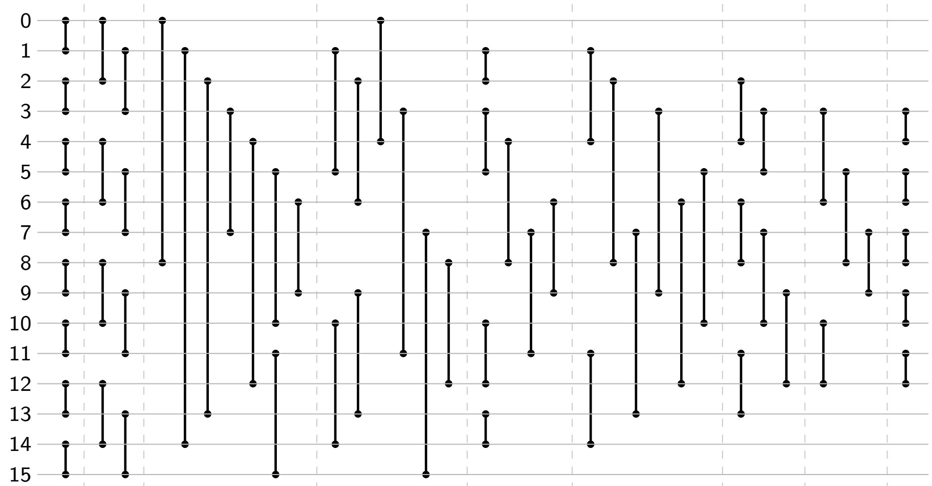
## Network D15  $61/9$, 2 bypasses



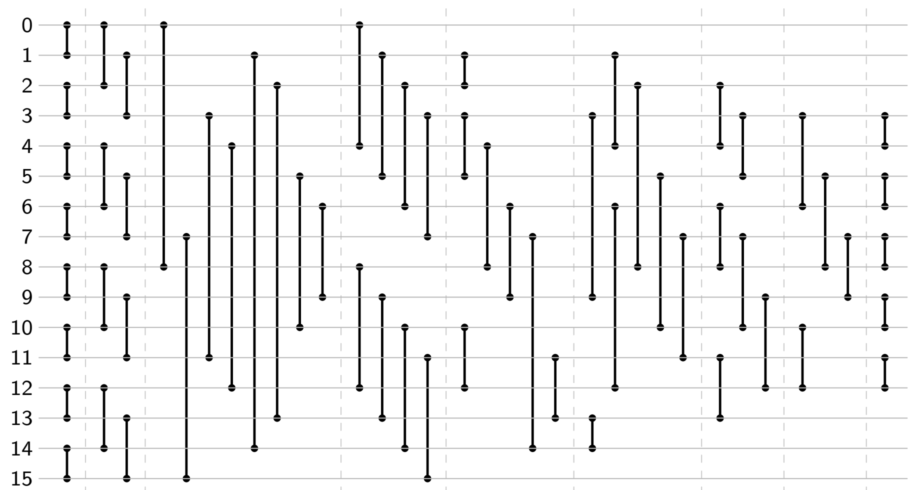

## Network D16  $61/9$, 2 bypasses



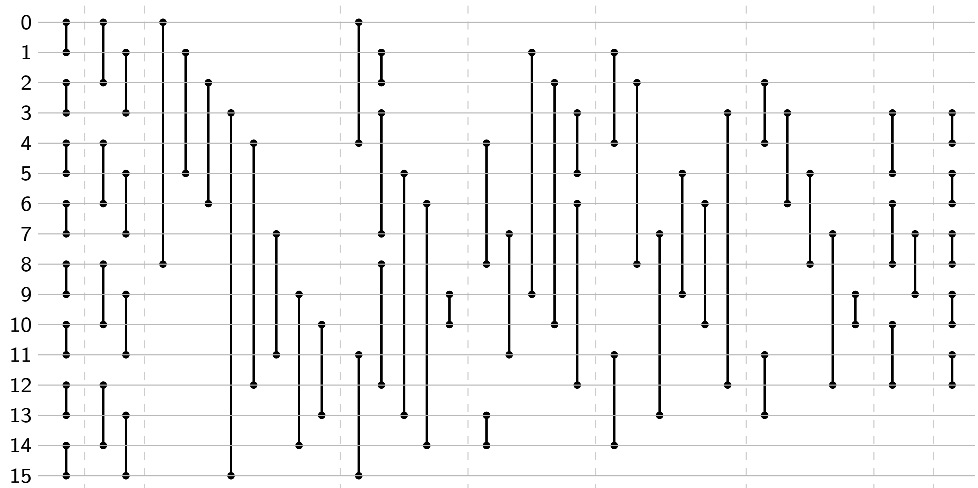
**Network D17** $61/9$, **2 bypasses**




**Network D18** $61/9$, **2 bypasses**



**Network D19** $61/9$, **2 bypasses**




**Network D20** $61/9$, **2 bypasses**